Notes on averaging over acyclic digraphs and discrete coverage control*

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Abstract

In this paper, we show the relationship between two algorithms and optimization problems that are the subject of recent attention in the networking and control literature. First, we obtain some results on averaging algorithms over acyclic digraphs with fixed and controlled-switching topology. Second, we discuss continuous and discrete coverage control laws. Further, we show how discrete coverage control laws can be cast as averaging algorithms defined over an appropriate graph that we term the discrete Voronoi graph.

1 Introduction

Consensus and coverage control are two distinct problems within the recent literature on multiagent coordination and cooperative robotics. Roughly speaking, the objective of the consensus problem is to analyze and design scalable distributed control laws to drive the groups of agents to agree upon certain quantities of interest. On the other hand, the objective of the coverage control problem is to deploy the agents to get optimal sensing performance of an environment of interest.

In the literature, many researchers have used averaging algorithms to solve consensus problems. The spirit of averaging algorithms is to let the state of each agent evolve according to the (weighted) average of the state of its neighbors. Averaging algorithms has been studied both in continuous time [2, 3, 4, 5] and in discrete time [6, 5, 7, 8, 9, 10, 11, 12]. In [2], averaging algorithms are investigated via graph Laplacians [13] under a variety of assumptions, including fixed and switching communication topologies, time delays, and directed and undirected information flow. In [3], a series of consensus protocols are presented, based on the regular averaging algorithms, to drive the agents to agree upon the value of the power mean. A theoretical explanation for the consensus behavior of the Vicsek model [14] is provided in [7], see also the early work in [6], while [5] extends the results of [7] to the case of directed topology for both continuous and discrete update schemes. The work [8] adopts a set-valued Lyapunov approach to analyze the convergence properties of averaging algorithms, which is generalized in [9] to the case of time delays. Asynchronous averaging algorithms are studied in [10]. The work [11] analyzes the averaging algorithms in the framework of partial difference equations over graphs [15]. The works [16, 17] survey the results available for consensus problems using averaging algorithms. In the scenario of coverage control, [18] proposes gradient descent algorithms for optimal coverage, and [19] presents coverage control algorithms for groups

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of mobile sensors with limited-range interactions. Also, we want to point out that a special kind of directed graphs, namely acyclic digraphs, are presented in the literature to describe the interactions of agents in leader-following formation problems, e.g., [20, 21, 22].

The contributions of this paper are (i) the investigation of the properties of averaging algorithms over acyclic digraphs with fixed and controlled-switching topologies, and (ii) the establishment of the connection between discrete coverage problems and averaging algorithms over acyclic digraphs. Regarding (i), our first contribution is a novel matrix representation of the disagreement function associated with a directed graph. Secondly, we prove that averaging over an fixed acyclic graph drives the agents to an equilibrium determined by the so-called "sinks" of the graph. Finally, we show that averaging over controlled-switching acyclic digraphs also makes the agents converge to the set of equilibria under suitable state-dependent switching signals. Regarding (ii), we present multicenter locational optimization functions in continuous and discrete settings, and discuss distributed coverage control algorithms that optimize them. We discuss how consistent discretizations of continuous coverage problems yield discrete coverage problems. Finally, we show how discrete coverage control laws over the discrete Voronoi graph can be casted and analyzed as averaging algorithms over a set of controlled-switching acyclic digraphs. Various simulations illustrate the results.

The paper is organized as follows. Section 2 introduces our novel matrix representation of the disagreement function, and then reviews the current results on consensus problems. We also present convergence results of averaging algorithms over acyclic digraphs with both fixed and controlled-switching topologies. Section 3 presents locational optimization functions in both continuous and discrete settings, and then discusses appropriate coverage control laws. The main result of the paper shows the relationship between averaging over switching acyclic digraphs and discrete coverage. Various simulations illustrate this result, and show the consistent parallelism between the continuous and the discrete settings. Finally, we gather our conclusions in Section 4. For easy reference, we review some basic facts and standard notations from nonsmooth analysis.

Notation

We let $\mathbb{N}$, $\mathbb{R}_+$ and $\overline{\mathbb{R}}_+$ denote, respectively, the set of natural numbers, the set of positive reals, and the set of non-negative reals. For any set $\mathcal{S} \subseteq \mathbb{R}^2$, we let $\overset{\circ}{\mathcal{S}}$, and $\partial \mathcal{S}$ denote, respectively, the interior, and boundary of $\mathcal{S}$. The quadratic form associated with a symmetric matrix $B \in \mathbb{R}^{n \times n}$ is the function defined by $x \mapsto x^T B x$. The map $f : X \to Y$ and the set-valued map $f : X \rightrightarrows Y$ associate to a point in X a point in Y and a subset of Y, respectively. The sum of m subsets $S_i, i \in \{1, \dots, m\}$ in a vector space, denoted by $\sum_{i=1}^m S_i$, consists of all vectors of the form $\sum_{i=1}^m s_i$, where $s_i \in S_i$, for $i \in \{1, \dots, m\}$.

2 Averaging algorithms over digraphs

2.1 Preliminaries on digraphs and disagreement functions

A *weighted directed graph*, in short *digraph*, $\mathcal{G} = (\mathcal{U}, \mathcal{E}, \mathcal{A})$ of order n consists of a *vertex set* $\mathcal{U}$ with n elements, an *edge set* $\mathcal{E} \in 2^{\mathcal{U} \times \mathcal{U}}$ (recall that $2^{\mathcal{U}}$ is the collection of subsets of $\mathcal{U}$), and a *weighted adjacency matrix* $\mathcal{A}$ with nonnegative entries a_{ij}, $i, j \in \{1, \dots, n\}$. For simplicity, we take $\mathcal{U} = \{1, \dots, n\}$. For $i, j \in \{1, \dots, n\}$, the entry a_{ij} is positive if and only if the pair (i, j) is an edge of $\mathcal{G}$, i.e., $a_{ij} > 0 \Leftrightarrow (i, j) \in \mathcal{E}$. We also assume $a_{ii} = 0$ for all $i \in \{1, \dots, n\}$ and $a_{ij} = 0$ if $(i, j) \notin \mathcal{E}$, for all $i, j \in \{1, \dots, n\}$ and $i \neq j$. When convenient, we will refer to the adjacency matrix of $\mathcal{G}$ by $\mathcal{A}(\mathcal{G})$.

Let us now review some basic connectivity notions for digraphs. A *directed path* in a digraph is an ordered sequence of vertices such that any two consecutive vertices in the sequence are an edge of the digraph. A *cycle* is a non-trivial directed path that starts and ends at the same vertex. A digraph is *acyclic* if it contains no directed cycles. A node of a digraph is *globally reachable* if it

can be reached from any other node by traversing a directed path. A digraph is *strongly connected* if every node is globally reachable.

Remark 2.1. *The previous definition of adjacency matrix follows the convention adopted in [2], where $a_{ij} > 0 \Leftrightarrow (i,j) \in \mathcal{E}$. On the other hand, in [16], $a_{ij} > 0 \Leftrightarrow (j,i) \in \mathcal{E}$. This difference arises from a different meaning of the direction of an edge. In [2], a directed edge $(i,j) \in \mathcal{E}$ means node i can 'see' node j, i.e., node i can obtain, in some way, information from node j. We refer to this as the* communication *interpretation. In [16], a directed edge $(i,j) \in \mathcal{E}$ means that the information of node i can flow to node j. We refer to this as the* sensing *interpretation. The difference leads to different statements of various results. For example, having a globally reachable node in the communication interpretation is equivalent to having a spanning tree in the sensing interpretation.* •

The *out-degree* and the *in-degree* of node i are defined by, respectively,

$$d_{\text{out}}(i) = \sum_{j=1}^{n} a_{ij}, \quad d_{\text{in}}(i) = \sum_{j=1}^{n} a_{ji}.$$

The out-degree matrix $D_{\text{out}}(\mathcal{G})$ and the in-degree matrix $D_{\text{in}}(\mathcal{G})$ are the diagonal matrices defined by $(D_{\text{out}}(\mathcal{G}))_{ii} = d_{\text{out}}(i)$ and $(D_{\text{in}}(\mathcal{G}))_{ii} = d_{\text{in}}(i)$, respectively. The digraph $\mathcal{G}$ is *balanced* if $D_{\text{out}}(\mathcal{G}) = D_{\text{in}}(\mathcal{G})$. The *graph Laplacian* of the digraph $\mathcal{G}$ is

$$L(\mathcal{G}) = D_{\text{out}}(\mathcal{G}) - \mathcal{A}(\mathcal{G}),$$

or, in components,

$$l_{ij}(\mathcal{G}) = \begin{cases} \displaystyle\sum_{k=1, k\neq i}^{n} a_{ik}, & j = i, \\ -a_{ij}, & j \neq i. \end{cases}$$

Next, we define reverse and mirror digraphs. Let $\tilde{\mathcal{E}}$ be the set of reverse edges of $\mathcal{G}$ obtained by reversing the order of all the pairs in $\mathcal{E}$. The *reverse digraph* of $\mathcal{G}$, denoted $\tilde{\mathcal{G}}$, is $(\mathcal{U}, \tilde{\mathcal{E}}, \tilde{\mathcal{A}})$, where $\tilde{\mathcal{A}} = \mathcal{A}^T$. The *mirror digraph* of $\mathcal{G}$, denoted $\hat{\mathcal{G}}$, is $(\mathcal{U}, \hat{\mathcal{E}}, \hat{\mathcal{A}})$, where $\hat{\mathcal{E}} = \mathcal{E} \cup \tilde{\mathcal{E}}$ and $\hat{\mathcal{A}} = (\mathcal{A} + \mathcal{A}^T)/2$. Note that $L(\tilde{\mathcal{G}}) = D_{\text{out}}(\tilde{\mathcal{G}}) - \mathcal{A}(\tilde{\mathcal{G}}) = D_{\text{in}}(\mathcal{G}) - \mathcal{A}(\mathcal{G})^T$.

Given a digraph $\mathcal{G}$ of order n, the *disagreement function* $\Phi_{\mathcal{G}} : \mathbb{R}^n \to \mathbb{R}$ is defined by

$$\Phi_{\mathcal{G}}(x) = \frac{1}{2} \sum_{i,j=1}^{n} a_{ij}(x_j - x_i)^2. \tag{1}$$

To the best of the authors' knowledge, the following is a novel result.

Proposition 2.2 (Matrix representation of disagreement)**.** *Given a digraph $\mathcal{G}$ of order n, the disagreement function $\Phi_{\mathcal{G}} : \mathbb{R}^n \to \mathbb{R}$ is the quadratic form associated with the symmetric positive-semidefinite matrix*

$$P(\mathcal{G}) = \frac{1}{2}(D_{\text{out}}(\mathcal{G}) + D_{\text{in}}(\mathcal{G}) - \mathcal{A}(\mathcal{G}) - \mathcal{A}(\mathcal{G})^T).$$

Moreover, $P(\mathcal{G})$ is the graph Laplacian of the mirror graph $\hat{\mathcal{G}}$, that is, $P(\mathcal{G}) = L(\hat{\mathcal{G}}) = \frac{1}{2}\Big(L(\mathcal{G}) + L(\tilde{\mathcal{G}})\Big)$.

Proof. For $x \in \mathbb{R}^n$, we compute

$$\begin{aligned} x^T P(\mathcal{G}) x &= \frac{1}{2} x^T (D_{\text{out}} + D_{\text{in}} - \mathcal{A} - \mathcal{A}^T) x \\ &= \frac{1}{2} \big(\sum_{i,j=1}^{n} a_{ij} x_i^2 + \sum_{i,j=1}^{n} a_{ij} x_j^2 - 2 \sum_{i,j=1}^{n} a_{ij} x_i x_j \big) \\ &= \frac{1}{2} \big(\sum_{i,j=1}^{n} a_{ij} (x_i^2 + x_j^2 - 2 x_i x_j) \big) \\ &= \frac{1}{2} \sum_{i,j=1}^{n} a_{ij} (x_j - x_i)^2 = \Phi_{\mathcal{G}}(x). \end{aligned}$$

Clearly P is symmetric. Since $\Phi_{\mathcal{G}}(x) \geq 0$ for all $x \in \mathbb{R}^n$, we deduce $P(\mathcal{G})$ is positive semidefinite. Since

$$(D(\hat{\mathcal{G}}))_{ii} = \sum_{j=1}^{n} \hat{a}_{ij} = \sum_{j=1}^{n} \frac{1}{2}(a_{ij} + a_{ji}),$$

we have $D(\hat{\mathcal{G}}) = \frac{1}{2}(D_{\text{out}}(\mathcal{G}) + D_{\text{in}}(\mathcal{G}))$. Hence,

$$\begin{aligned} L(\hat{\mathcal{G}}) &= D(\hat{\mathcal{G}}) - \mathcal{A}(\hat{\mathcal{G}}) \\ &= \frac{1}{2}(D_{\text{out}}(\mathcal{G}) + D_{\text{in}}(\mathcal{G})) - \frac{1}{2}(\mathcal{A}(\mathcal{G}) + \mathcal{A}(\mathcal{G})^T) = P(\mathcal{G}). \end{aligned}$$

The last inequality follows from the definitions of reverse and mirror graphs.

Remark 2.3. *Note that in general, $P(\mathcal{G}) \neq L(\mathcal{G})$. However, if the digraph $\mathcal{G}$ is balanced, then $D_{\text{out}}(\mathcal{G}) = D_{\text{in}}(\mathcal{G})$, and therefore,*

$$\begin{aligned} \Phi_{\mathcal{G}}(x) &= \frac{1}{2} x^T (D_{\text{out}}(\mathcal{G}) + D_{\text{in}}(\mathcal{G})) x - \frac{1}{2} x^T (\mathcal{A}(\mathcal{G}) + \mathcal{A}(\mathcal{G})^T) x \\ &= x^T D_{\text{out}}(\mathcal{G}) x - x^T \mathcal{A} x = x^T L(\mathcal{G}) x. \end{aligned}$$

This is the result usually presented in the literature on undirected graphs. •

2.2 Averaging plus connectivity achieves consensus

To each node $i \in \mathcal{U}$ of a digraph $\mathcal{G}$, we associate a state $x_i \in \mathbb{R}$, that obeys a first-order dynamics of the form

$$\dot{x}_i = u_i, \quad i \in \{1, \ldots, n\}.$$

We say that the nodes of a network have reached a *consensus* if $x_i = x_j$ for all $i, j \in \{1, \ldots, n\}$. Our objective is to design control laws u that guarantee that consensus is achieved starting from any initial condition, while u_i depends only on the state of the node i and of its neighbors in $\mathcal{G}$, for $i \in \{1, \ldots, n\}$. In other words, the closed-loop system asymptotically achieves consensus if, for any $x_0 \in \mathbb{R}^n$, one has that $x(t) \to \{\alpha(1, \ldots, 1) \mid \alpha \in \mathbb{R}\}$ when $t \to +\infty$. If the value α is the average of the initial state of the n nodes, then we say the nodes have reached *average-consensus*.

We refer to the following linear control law, often used in the literature on consensus (e.g., see [7, 10, 16]), as the *averaging protocol*:

$$u_i = \sum_{j=1}^{n} a_{ij}(x_j - x_i). \tag{2}$$

With this control law, the closed-loop system is

$$\dot{x}(t) = -L(\mathcal{G})x(t). \tag{3}$$

Next, we consider a family of digraphs $\{\mathcal{G}_1, \ldots, \mathcal{G}_m\}$ with the same vertex set $\{1, \ldots, n\}$. A *switching signal* is a map $\sigma : \overline{\mathbb{R}}_+ \times \mathbb{R}^n \to \{1, \ldots, m\}$. Given these objects, we can define the following switched dynamical system

$$\begin{aligned} \dot{x}(t) &= -L(\mathcal{G}_k)x(t), \\ k &= \sigma(t, x(t)). \end{aligned} \tag{4}$$

Note that the notion of solution for this system might not be well-defined for arbitrary switching signals. The properties of the linear system (3) and the system (4) under time-dependent switching signals have been investigated in [2, 5, 8, 23]. Here, we review some of these properties in the following two statements.

Theorem 2.4 (Averaging over a digraph)**.** *Let $\mathcal{G}$ be a digraph. The following statements hold:*

(i) System (3) *asymptotically achieves consensus if and only if $\mathcal{G}$ has a globally reachable node;*

(ii) If $\mathcal{G}$ is strongly connected, then system (3) *asymptotically achieves average-consensus if and only if $\mathcal{G}$ is balanced.*

Statement (i) is proved in [23, Section 2]. Statement (ii) is proved in [2, Section VII].

Theorem 2.5 (Averaging over switching digraphs)**.** *Let $\{\mathcal{G}_1, \ldots, \mathcal{G}_m\}$ be a family of digraphs with the same vertex set $\{1, \ldots, n\}$, and let $\sigma : \overline{\mathbb{R}}_+ \to \{1, \ldots, m\}$ be a piecewise constant function. The following statements hold:*

(i) System (4) *asymptotically achieves consensus if there exist infinitely many consecutive uniformly bounded time intervals such that the union of the switching graphs across each interval has a globally reachable node;*

(ii) If each $\mathcal{G}_i$, $i \in \{1, \ldots, m\}$, is strongly connected and balanced, then for any arbitrary piecewise constant function σ, the system (4) *globally asymptotically solves the* average-*consensus problem.*

Statement (i) is proved in [5, Section III B]. Statement (ii) is proved in [2, Section IX].

2.3 Averaging protocol over a fixed acyclic digraph

Here we characterize the convergence properties of the averaging protocol in equation (3) under different connectivity properties than the ones stated in Theorem 2.4, namely assuming that the given digraph is acyclic.

We start by reviewing some basic properties of acyclic digraphs. Given an acyclic digraph $\mathcal{G}$, every vertex of in-degree 0 is named *source*, and every vertex of out-degree 0 is named *sink*. Every acyclic digraph has at least one source and at least one sink. (Recall that sources and sinks can be identified by following any directed path on the digraph.) Given an acyclic digraph $\mathcal{G}$, we associate a nonnegative number to each vertex, called *depth*, in the following way. First, we define the depth of the sinks of $\mathcal{G}$ to be 0. Next, we consider the acyclic digraph that results from erasing the 0-depth vertices from $\mathcal{G}$ and the in-edges towards them; the depth of the sinks of this new acyclic digraph are defined to be 1. The higher depth vertices are defined recursively. This process is well-posed as any acyclic digraph has at least one sink. The depth of the digraph is the maximum depth of its vertices. For $n, d \in \mathbb{N}$, let $\mathcal{S}_{n,d}$ be the set of acyclic digraphs with vertex set $\{1, \ldots, n\}$ and depth d.

Next, it is convenient to relabel the n vertices of the acyclic digraph $\mathcal{G}$ with depth d in the following way: (1) label the sinks from 1 to n_0, where n_0 is the number of sinks; (2) label the

vertices of depth k from $\sum_{j=0}^{k-1} n_j + 1$ to $\sum_{j=0}^{k-1} n_j + n_k$, where n_k is the number of vertices of depth k, for $k \in \{1, \dots, d\}$. Note that vertices with the same depth may be labeled in arbitrary order. With this labeling, the adjacency matrix $\mathcal{A}(\mathcal{G})$ is lower-diagonal with vanishing diagonal entries, and the Laplacian $L(\mathcal{G})$ takes the form

$$L(\mathcal{G}) = \begin{bmatrix} 0 & 0 & \dots & 0 \\ -a_{21} & \sum_{j=1}^{1} a_{2j} & \dots & 0 \\ \dots & \dots & \dots & \dots \\ -a_{n1} & -a_{n2} & \dots & \sum_{j=1}^{n-1} a_{nj} \end{bmatrix},$$

or, alternatively,

$$L(\mathcal{G}) = \begin{bmatrix} 0_{n_0 \times n_0} & 0_{n_0 \times (n-n_0)} \\ L_{21} & L_{22} \end{bmatrix}, \tag{5}$$

where $0_{k\times h}$ is the $k \times h$ matrix with vanishing entries, $L_{21} \in \mathbb{R}^{(n-n_0)\times n_0}$ and $L_{22} \in \mathbb{R}^{n-n_0\times n-n_0}$. Clearly, all eigenvalues of L are non-negative and the zero eigenvalues are simple, as their corresponding Jordan blocks are 1×1 matrices.

Proposition 2.6 (Averaging over an acyclic digraph)**.** *Let $\mathcal{G}$ be an acyclic digraph of order n with n_0 sinks, assume its vertices are labeled according to their depth, and consider the dynamical system $\dot{x}(t) = -L(\mathcal{G})x(t)$ defined in* (3)*. The following statements hold:*

(i) The equilibrium set of (3) *is the vector subspace*

$$\ker L(\mathcal{G}) = \{(x_{\mathrm{s}}, x_{\mathrm{e}}) \in \mathbb{R}^{n_0} \times \mathbb{R}^{n-n_0} \mid x_{\mathrm{e}} = -L_{22}^{-1} L_{21} x_{\mathrm{s}}\}.$$

(ii) Each trajectory $x : \overline{\mathbb{R}}_+ \to \mathbb{R}^n$ of (3) *exponentially converges to the equilibrium x^* defined recursively by*

$$x_i^* = \begin{cases} x_i(0), & i \in \{1, \dots, n_0\}, \\ \dfrac{\sum_{j=1}^{i-1} a_{ij} x_j^*}{\sum_{j=1}^{i-1} a_{ij}}, & i \in \{n_0 + 1, \dots, n\}. \end{cases}$$

(iii) If $\mathcal{G}$ has unit depth, then the disagreement function $\Phi_{\mathcal{G}}$ is monotonically decreasing along any trajectory of (3).

Proof. Statement (i) is obvious. Statement (ii) follows from the fact that $-L_{22}$ is Hurwitz and from the equilibrium equality

$$0 = \sum_{j=1}^{i-1} a_{ij}(x_j^* - x_i^*) = \sum_{j=1}^{i-1} a_{ij} x_j^* - \Big(\sum_{j=1}^{i-1} a_{ij}\Big) x_i^*.$$

Regarding statement (iii), when the depth of $\mathcal{G}$ is 1, the adjacency matrix and the out-degree matrix are equal to, respectively,

$$\begin{bmatrix} 0_{n_0\times n_0} & 0_{n_0\times(n-n_0)} \\ -L_{21} & 0_{(n-n_0)\times(n-n_0)} \end{bmatrix}, \begin{bmatrix} 0_{n_0\times n_0} & 0_{n_0\times(n-n_0)} \\ 0_{(n-n_0)\times n_0} & L_{22} \end{bmatrix},$$

where L_{21} and L_{22} are defined in (5). Therefore, we compute

$$L(\tilde{\mathcal{G}}) = \begin{bmatrix} \tilde{L}_{11} & L_{21}^T \\ 0_{(n-n_0)\times n_0} & 0_{(n-n_0)\times(n-n_0)} \end{bmatrix},$$

where $\tilde{L}_{11} \in \mathbb{R}^{n_0\times n_0}$. According to Proposition 2.2, we have

$$P(\mathcal{G}) = \frac{1}{2}\Big(L(\mathcal{G}) + L(\tilde{\mathcal{G}})\Big)$$

The evolution of $\Phi_{\mathcal{G}}$ along a trajectory of $x : \overline{\mathbb{R}}_+ \to \mathbb{R}^n$ of (3) is given by

$$\begin{aligned}\frac{d}{dt}\big(\Phi_{\mathcal{G}}(x(t))\big) &= -x(t)^T(L(\mathcal{G})^T P(\mathcal{G}) + P(\mathcal{G})L(\mathcal{G}))x(t)\\ &= -x(t)^T L(\mathcal{G})^T L(\mathcal{G})x(t) - x(t)^T L(\mathcal{G})^T L(\tilde{\mathcal{G}})x(t)\\ &= -x(t)^T L(\mathcal{G})^T L(\mathcal{G})x(t) \le 0,\end{aligned}$$

where in the last equality we have used the fact that $L(\mathcal{G})^T L(\tilde{\mathcal{G}}) = L(\tilde{\mathcal{G}})^T L(\mathcal{G}) = 0_{n\times n}$. Note that $\Phi_{\mathcal{G}}$ is strictly decreasing unless $x(t) \in \ker L(\mathcal{G})$, i.e., the trajectory reaches an equilibrium.

Remarks 2.7. *(i) If the digraph has a single sink, then the convergence statement in part (ii) of Proposition 2.6 is equivalent to part (i) of Theorem 2.4.*

(ii) The block decomposition of $L(\tilde{\mathcal{G}})$ holds only for digraphs with depth 1. Indeed, statement (iii) is not true for digraphs with depth larger than 1. The digraph in Figure 1 is a counterexample.

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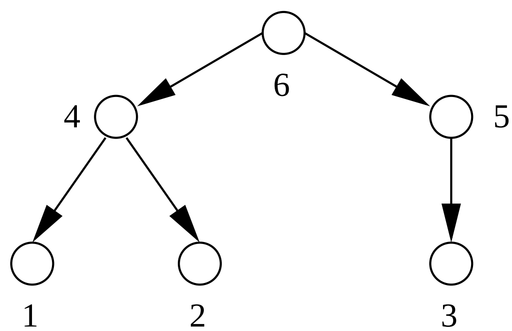

Figure 1: For this digraph of depth 2, the Lie derivative of the disagreement function (1) along the averaging flow (3) is indefinite.

2.4 Averaging protocol over switching acyclic digraphs

Given a family of digraphs $\Gamma = \{\mathcal{G}_1, \dots, \mathcal{G}_m\}$ with vertex set $\{1, \dots, n\}$, the *minimal disagreement function* $\Phi_\Gamma : \mathbb{R}^n \to \mathbb{R}$ is defined by

$$\Phi_\Gamma(x) = \min_{k\in\{1,\dots,m\}} \Phi_{\mathcal{G}_k}(x). \tag{6}$$

Let $I(x) = \operatorname{argmin}\{\Phi_{\mathcal{G}_k}(x) \mid k \in \{1, \dots, m\}\}$, we consider state-dependent switching signals $\sigma : \mathbb{R}^n \to \{1, \dots, m\}$ with the property that $\sigma(x) \in I(x)$, that is, at each $x \in \mathbb{R}^n$, $\sigma(x)$ corresponds to the index of a graph with minimal disagreement. Clearly, for any such σ, one has $\Phi_\Gamma(x) = \Phi_{\mathcal{G}_{\sigma(x)}}(x)$. Before giving our result, we first point out a helpful fact.

Lemma 2.8. *Let $\Gamma = \{\mathcal{G}_1, \dots, \mathcal{G}_m\} \subset \mathcal{S}_{n,1}$. If $\cup_{k\in\{1,\dots,m\}} \mathcal{G}_k \in \mathcal{S}_{n,1}$, then for any $i, j \in \{1, \dots, m\}$, we have*

$$L(\mathcal{G}_i)^T L(\tilde{\mathcal{G}}_j) = 0_{n\times n}.$$

Proof. Let $\mathcal{G} = \cup_{k\in\{1,\dots,m\}} \mathcal{G}_k$. Since $\mathcal{G} \in \mathcal{S}_{n,1}$, so we have, by proper ordering of the nodes,

$$L(\mathcal{G}) = \begin{bmatrix} 0_{n_0\times n_0} & 0_{n_0\times(n-n_0)} \\ L_{21} & L_{22}\end{bmatrix},\ L(\tilde{\mathcal{G}}) = \begin{bmatrix} \tilde{L}_{11} & L_{21}^T \\ 0_{(n-n_0)\times n_0} & 0_{(n-n_0)\times(n-n_0)}\end{bmatrix}.$$

For any $i \in \{1, \dots, m\}$, $\mathcal{G}_i$ is a subgraph of $\mathcal{G}$, so that $L(\mathcal{G}_i)$ and $L(\tilde{\mathcal{G}}_i)$ share the same block decompositions as stated in the last equation. Hence, the statement follows immediately.

Proposition 2.9 (Averaging over acyclic digraphs)**.** *Let $\Gamma = \{\mathcal{G}_1, \dots, \mathcal{G}_m\} \subset \mathcal{S}_{n,1}$, and assume that $\cup_{k\in\{1,\dots,m\}} \mathcal{G}_k \in \mathcal{S}_{n,1}$ and that $\sigma : \mathbb{R}^n \to \{1, \dots, m\}$ satisfies $\sigma(x) \in I(x)$. Consider the discontinuous dynamical system*

$$\dot{x}(t) = Y(x(t)) = -L(\mathcal{G}_k)x(t), \quad \text{for} \quad k = \sigma(x(t)), \tag{7}$$

whose solutions are understood in the Filippov sense. The following statements hold:

(i) The point $x^* \in \mathbb{R}^n$ is an equilibrium for (7) if and only if for each $i \in I(x^*)$, there exists scalars $\lambda_i \geq 0$ and $\sum_{i\in I(x^*)} \lambda_i = 1$, such that

$$x^* \in \ker\Big(\sum_{i\in I(x^*)} \lambda_i L(\mathcal{G}_i)\Big). \tag{8}$$

In particular, if $I(x^*)$ contains only one element $k^* \in \{1, \dots, m\}$, then (8) can be simplified to

$$x^* \in \ker L(\mathcal{G}_{k^*}). \tag{9}$$

(ii) Each trajectory $x : \overline{\mathbb{R}}_+ \to \mathbb{R}^n$ of (7) converges to the set of equilibria.

(iii) The minimum disagreement function Φ_Γ is monotonically non-increasing along any trajectory $x : \overline{\mathbb{R}}_+ \to \mathbb{R}^n$ of (7).

Proof. We investigate first smoothness of Φ_Γ. Because $-\Phi_\Gamma$ is the maximum of the smooth functions $-\Phi_{\mathcal{G}_k}$, by Proposition A.3, we know that Φ_Γ is locally Lipschitz and has generalized gradient

$$\partial\Phi_\Gamma(x) = \mathrm{co}\{2P(\mathcal{G}_i)x \mid i \in I(x)\}.$$

Let $a \in \widetilde{\mathcal{L}}_Y \Phi_\Gamma(x)$, then by definition, there exists $\omega = -\sum_{i\in I(x)} \lambda_i L(\mathcal{G}_i)x$, where, for each $i \in I(x)$, $\lambda_i \geq 0$ and $\sum_{i\in I(x)} \lambda_i = 1$, such that $a = \omega^T\zeta$ for all $\zeta \in \partial\Phi_\Gamma(x)$. In particular, for $\zeta = \sum_{i\in I(x)} 2\lambda_i P(\mathcal{G}_i)x \in \partial\Phi_\Gamma(x)$, we have

$$\begin{aligned}
a &= \Big(-\sum_{i\in I(x)} \lambda_i L(\mathcal{G}_i)x\Big)^T \Big(\sum_{i\in I(x)} 2\lambda_i P(\mathcal{G}_i)x\Big) \\
&= -x^T\Big(\sum_{i\in I(x)} \lambda_i L(\mathcal{G}_i)\Big)^T \Big(\sum_{i\in I(x)} \lambda_i (L(\mathcal{G}_i) + L(\tilde{\mathcal{G}}_i))\Big)x \\
&= -x^T\Big(\sum_{i\in I(x)} \lambda_i L(\mathcal{G}_i)\Big)^T \Big(\sum_{i\in I(x)} \lambda_i L(\mathcal{G}_i)\Big)x - x^T\Big(\sum_{i\in I(x)} \lambda_i L(\mathcal{G}_i)^T\Big)\Big(\sum_{i\in I(x)} \lambda_i L(\tilde{\mathcal{G}}_i)\Big)x \\
&= -x^T\Big(\sum_{i\in I(x)} \lambda_i L(\mathcal{G}_i)\Big)^T \Big(\sum_{i\in I(x)} \lambda_i L(\mathcal{G}_i)\Big)x \leq 0,
\end{aligned}$$

where in the last equality we have used Lemma 2.8. Moreover,

$$a = 0 \quad \Longleftrightarrow \quad x \in \ker\Big(\sum_{i\in I(x)} \lambda_i L(\mathcal{G}_i)\Big).$$

In particular, if x is not at any switching surface, then $I(x)$ is a set with only one element $k \in \{1, \dots, m\}$ and $\partial\Phi_\Gamma(x) = 2P(\mathcal{G}_k)x$. Therefore, $\widetilde{\mathcal{L}}_Y \Phi_\Gamma(x) = 0$ if and only if $x \in \ker L(\mathcal{G}_k)$. Therefore, we conclude that for $x \in \mathbb{R}^n$ and $a \in \widetilde{\mathcal{L}}_Y \Phi_\Gamma(x)$, we have $a \leq 0$, i.e., $\max \widetilde{\mathcal{L}}_Y \Phi_\Gamma(x) \leq 0$ and statement (i) is true. Resorting to the LaSalle Invariance Principle (Theorem B.2), we deduce that any trajectory $x : \overline{\mathbb{R}}_+ \to \mathbb{R}^n$ of (7) converges to the set of equilibria as stated in statement (i) and statement (iii) is clear.

Remarks 2.10.
- *Statement (ii) in this theorem is weaker than statement (ii) in previous one in three ways: first, we are not able to characterize the limit point as a function of the initial state. Second, we require the depth 1 assumption, which is sufficient to ensure convergence, but possibly not necessary. Third, we establish only convergence to a set, rather than an individual point. It remains an open question to obtain necessary and sufficient conditions for convergence to a point.*
- *Although the statement (ii) is obtained only for digraphs of unit depth, this class of graphs is of interest in the forthcoming sections.* •

3 Discrete coverage control

In this section, we first review the multi-center optimization problem and the corresponding coverage control algorithm proposed in [18]. We then study the multi-center optimization problem in discrete space and derive a discrete coverage control law. This leads to a geometric object called the discrete Voronoi graph. Finally, we show that the discrete coverage control law is an averaging algorithm over a certain set of acyclic digraphs. Discrete locational optimization problems are discussed in [24, 25, 26].

We will consider motion coordination problems for a group of robots described by first order integrators. In other words, we assume that n robotic agents are placed at locations $p_1, \dots, p_n \in \mathbb{R}^2$ and that they move according to

$$\dot{p}_i = u_i, \quad i \in \{1, \dots, n\}. \tag{10}$$

We denote by P the vector of positions $(p_1, \dots, p_n) \in (\mathbb{R}^2)^n$. Additionally, we define

$$\mathcal{S}_{\text{coinc}} = \{(p_1, \dots, p_n) \in (\mathbb{R}^2)^n \mid p_i = p_j \text{ for some } i \neq j\},$$

and, for $P \notin \mathcal{S}_{\text{coinc}}$, we let $\{V_i(P)\}_{i \in \{1,\dots,n\}}$ denote the Voronoi partition generated by P, we illustrate this notion in Figure 2 and refer to [24] for a comprehensive treatment on Voronoi partitions.

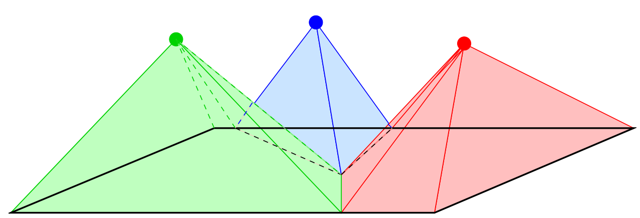

Figure 2: The Voronoi partition of a rectangle in the plane. We depict the generators $p_1, \dots, p_n$ elevated from the plane for intuition's sake.

3.1 Continuous and discrete multi-center functions

In this section we present a class of locational optimization problems in both continuous and discrete settings. It would be possible to provide a unified treatment using generalized functions and distributions, but we avoid it here for concreteness' sake.

Let Q be a convex polygon in $\mathbb{R}^2$ including its interior and let $\phi : \mathbb{R}^2 \to \overline{\mathbb{R}}_+$ be a bounded and measurable function whose support is Q. Analogously, let $\{q_1, \dots, q_N\} \subset \mathbb{R}^2$ be a pointset and $\{\phi_1, \dots, \phi_N\}$ be nonnegative weights associated to them. Given a non-increasing function $f : \overline{\mathbb{R}}_+ \to \mathbb{R}$, we consider the *continuous* and *discrete multi-center functions* $\mathcal{H} : (\mathbb{R}^2)^n \to \mathbb{R}$ and

$\mathcal{H}_{\text{dscrt}} : (\mathbb{R}^2)^n \to \mathbb{R}$ defined by

$$\mathcal{H}(P) = \int_Q \max_{i\in\{1,\dots,n\}} f(\|q-p_i\|)\phi(q)dq,$$
$$\mathcal{H}_{\text{dscrt}}(P) = \sum_{j=1}^{N} \max_{i\in\{1,\dots,n\}} \phi_j f(\|q_j-p_i\|).$$

Now we define

$$\mathcal{S}_{\text{equid}} = \{(p_1,\dots,p_n) \in (\mathbb{R}^2)^n \mid \|q-p_i\| = \|q-p_k\| = d(q) \quad \text{for some } q \in \{q_1,\dots,q_N\} \text{ and for some } i \neq k\},$$

where $d(q) = \min_{j\in\{1,\dots,n\}} \|q-p_j\|$. In other words, if $P \notin \mathcal{S}_{\text{equid}}$, then no point q_j is equidistant to two or more nearest robots. Note that $\mathcal{S}_{\text{equid}}$ is a set of measure zero because it is the union of the solutions of a finite number of algebraic equations. Using Voronoi partitions, for $P \notin \mathcal{S}_{\text{coinc}}$, we may write

$$\begin{aligned}
\mathcal{H}(P) &= \sum_{i=1}^{n} \int_{V_i(P)} f(\|q-p_i\|)\phi(q)dq,\\
\mathcal{H}_{\text{dscrt}}(P) &= \sum_{i=1}^{n} \sum_{q_j\in V_i(P)} \frac{\phi_j}{\operatorname{card}(q_j,P)} f(\|q_j-p_i\|)\\
&= \sum_{i=1}^{n} \left(\sum_{q_j\in \overset{\circ}{V}_i(P)} \phi_j f(\|q_j-p_i\|) + \sum_{q_j\in\partial V_i(P)} \frac{\phi_j}{\operatorname{card}(q_j,P)} f(\|q_j-p_i\|) \right),
\end{aligned}$$

where $\operatorname{card} : \mathbb{R}^2 \times (\mathbb{R}^2)^n \to \{1,\dots,n\}$ denotes the number of indices k for which $\|q_j-p_k\| = \min_{i\in\{1,\dots,n\}} \|q_j-p_i\|$. It is easy to see that card is distributed over the Voronoi graph and if q_j is a point in the interior of $V_i(P)$, then $\operatorname{card}(q_j,P)=1$. For $P \notin \mathcal{S}_{\text{coinc}} \cup \mathcal{S}_{\text{equid}}$, we have

$$\mathcal{H}_{\text{dscrt}}(P) = \sum_{i=1}^{n} \sum_{q_j\in \overset{\circ}{V}_i(P)} \phi_j f(\|q_j-p_i\|).$$

Remark 3.1. *The function f plays the role of a performance function. If $\{p_1,\dots,p_n\}$ are the locations of n sensors, and if events take place inside the environment Q with likelihood ϕ, then $f(\|q-p_i\|)$ is the quality of service provided by sensor i. It will therefore be of interest to find local maxima for $\mathcal{H}$ and $\mathcal{H}_{\text{dscrt}}$. These types of optimal sensor placement spatial resource allocation problems are the subject of a discipline called locational optimization [24, 25, 18].* •

The following result is discussed in[19] for the continuous multi-center function.

Proposition 3.2 (Partial derivatives of $\mathcal{H}$)**.** *If f is locally Lipschitz, then $\mathcal{H}$ is locally Lipschitz on Q^n. Further, if f is differentiable, then $\mathcal{H}$ is differentiable on $Q^n \setminus \mathcal{S}_{\text{coinc}}$, and, for each $i \in \{1,\dots,n\}$,*

$$\frac{\partial \mathcal{H}}{\partial p_i}(P) = \int_{V_i(P)} \frac{\partial}{\partial p_i} f(\|q-p_i\|)\phi(q)dq.$$

We obtain the corresponding properties of $\mathcal{H}_{\text{dscrt}}$ via nonsmooth analysis as the following proposition.

Proposition 3.3 (Generalized gradient of $\mathcal{H}_{\mathrm{dscrt}}$)**.** *If f is locally Lipschitz, then $\mathcal{H}_{\mathrm{dscrt}}$ is locally Lipschitz on Q^n. Further, if f is differentiable, then $\mathcal{H}_{\mathrm{dscrt}}$ is regular on Q^n and its generalized gradient satisfies*

$$\partial \mathcal{H}_{\mathrm{dscrt}}(P) = \sum_{j=1}^{N} \phi_j \operatorname{co}\Big\{ \frac{\partial}{\partial P} f(\|q_j - p_k\|) \mid k \in I(q_j, P) \Big\},$$

where $I(q_j, P)$ is the set of indices k for which $f(\|q_j - p_k\|) = \max_{i\in\{1,\dots,n\}} f(\|q_j - p_i\|)$, and in particular, if $P \notin \mathcal{S}_{\mathrm{coinc}} \cup \mathcal{S}_{\mathrm{equid}}$, then $\mathcal{H}_{\mathrm{dscrt}}$ is differentiable at P, and for each $i \in \{1, \dots, n\}$

$$\frac{\partial}{\partial p_i} \mathcal{H}_{\mathrm{dscrt}}(P) = \sum_{q_j \in \overset{\circ}{V}_i(P)} \phi_j \frac{\partial}{\partial p_i} f(\|q_j - p_i\|).$$

Proof. We re-write here $\mathcal{H}_{\mathrm{dscrt}}$ as

$$\mathcal{H}_{\mathrm{dscrt}}(P) = \sum_{j=1}^{N} \max_{i\in\{1,\dots,n\}} \phi_j f(\|q_j - p_i\|) = \sum_{j=1}^{N} \phi_j F_j(P),$$

where, for each $j \in \{1, \dots, N\}$,

$$F_j(P) = \max_{i\in\{1,\dots,n\}} f(\|q_j - p_i\|).$$

We first investigate the smoothness of $F_j(P)$. By Proposition A.3, it is easy to see that if f is locally Lipschitz, then $F_j(P)$ is locally Lipschitz on Q^n, so is $\mathcal{H}_{\mathrm{dscrt}}(P)$.

Additionally, if f is differentiable, then $F_j(P)$ is regular on Q^n, with generalized derivative

$$\partial F_j(P) = \operatorname{co}\Big\{ \frac{\partial}{\partial P} f(\|q_j - p_i\|) \mid i \in I(q_j, P) \Big\},$$

where $I(q_j, P)$ is the set of indexes k for which $f(\|q_j - p_k\|) = F_j(P)$. Since $\mathcal{H}_{\mathrm{dscrt}}(P)$ is a finite sum of $F_j(P)$ with nonnegative weights ϕ_j, so $\mathcal{H}_{\mathrm{dscrt}}(P)$ is also regular (cf. Proposition A.2) on Q^n, with the regularity of $F_j(P)$, we obtain further the generalized gradient of $\mathcal{H}_{\mathrm{dscrt}}(P)$ as

$$\partial \mathcal{H}_{\mathrm{dscrt}}(P) = \sum_{j=1}^{N} \phi_j \operatorname{co}\Big\{ \frac{\partial}{\partial P} f(\|q_j - p_k\|) \mid k \in I(q_j, P) \Big\}.$$

The expression for the partial derivative away from $\mathcal{S}_{\mathrm{coinc}} \cup \mathcal{S}_{\mathrm{equid}}$ is easy to see.

Let $\partial_i \mathcal{H}_{\mathrm{dscrt}}(P)$ denote the i^{th} block component of $\partial \mathcal{H}_{\mathrm{dscrt}}(P)$, the following result is a consequence of Proposition 3.3.

Corollary 3.4. *If f is differentiable, then for each $i \in \{1, \dots, n\}$,*

$$\partial_i \mathcal{H}_{\mathrm{dscrt}}(P) \subset \sum_{q_j \in \overset{\circ}{V}_i(P)} \phi_j \frac{\partial}{\partial p_i} f(\|q_j - p_i\|) + \sum_{q_j \in \partial V_i(P)} \phi_j \operatorname{co}\Big\{ \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \frac{\partial}{\partial p_i} f(\|q_j - p_i\|) \Big\}.$$

For particular choices of f, the multi-center functions and their partial derivatives may simplify. For example, if $f(x) = -x^2$, the partial derivative of the multi-center function $\mathcal{H}$ reads (for $P \notin \mathcal{S}_{\mathrm{coinc}}$)

$$\frac{\partial \mathcal{H}}{\partial p_i}(P) = 2M_{V_i(P)}(C_{V_i(P)} - p_i),$$

where mass and the centroid of $W \subset Q$ are

$$M_W = \int_W \phi(q)\, dq, \quad C_W = \frac{1}{M_W} \int_W q\, \phi(q)\, dq.$$

Additionally, the critical points P^* of $\mathcal{H}$ have the property that $p_i^* = C_{V_i(P^*)}$, for $i \in \{1, \dots, n\}$; these are called *centroidal Voronoi configurations*. Analogously, if $f(x) = -x^2$, the discrete multi-center function $\mathcal{H}_{\text{dscrt}}$ reads

$$\mathcal{H}_{\text{dscrt}}(P) = -\sum_{j=1}^{N} \max_{i \in \{1,\dots,n\}} \phi_j \|q_j - p_i\|^2,$$

and its generalized gradient is

$$\partial \mathcal{H}_{\text{dscrt}}(P) = \sum_{j=1}^{N} \phi_j \operatorname{co} \Big\{ 2(q_j - p_k) \frac{\partial p_k}{\partial P} \mid k \in I(q_j, P) \Big\}. \tag{11}$$

For each $j \in \{1, \dots, N\}$, assume the scalars λ_{ij}, $i \in I(q_j, P)$, satisfy

$$\lambda_{ij} \geq 0, \qquad \sum_{i \in I(q_j,P)} \lambda_{ij} = 1. \tag{12}$$

Next, define $(M_{\text{dscrt}})_{V_i(P)}$ and $(C_{\text{dscrt}})_{V_i(P)}$, respectively, as

$$(M_{\text{dscrt}})_{V_i(P)} = \sum_{q_j \in \overset{\circ}{V}_i(P)} \phi_j + \sum_{q_j \in \partial V_i(P)} \lambda_{ij} \phi_j = \sum_{q_j \in V_i(P)} \lambda_{ij} \phi_j,$$

$$(C_{\text{dscrt}})_{V_i(P)} = \begin{cases} p_i, & \text{if } (M_{\text{dscrt}})_{V_i(P)} = 0, \\ \dfrac{1}{(M_{\text{dscrt}})_{V_i(P)}} \Big(\displaystyle\sum_{q_j \in V_i(P)} \lambda_{ij} \phi_j q_j \Big), & \text{otherwise.} \end{cases}$$

Lemma 3.5. *Given $f(x) = -x^2$, P^* is a critical point of $\partial H_{\text{dscrt}}$, i.e., $0 \in \partial H_{\text{dscrt}}(P^*)$, if and only if for any $j \in \{1, \dots, N\}$, there exist λ_{ij} as in (12), such that $p_i^* = (C_{\text{dscrt}})_{V_i(P^*)}$, for each $i \in \{1, \dots, n\}$.*

Proof. Given scalar satisfying (12), define

$$w = \sum_{j=1}^{N} \phi_j \sum_{k \in I(q_j,P)} 2\lambda_{kj} (q_j - p_k^*) \frac{\partial p_k}{\partial P},$$

then it is clear that $w \in \partial \mathcal{H}_{\text{dscrt}}(P^*)$. Let w_i denotes the i^{th} component of w, since

$$w_i = 2 \sum_{q_j \in V_i(P^*)} \lambda_{ij} \phi_j (q_j - p_i^*) = 2 \sum_{q_j \in V_i(P^*)} \lambda_{ij} \phi_j (q_j - (C_{\text{dscrt}})_{V_i(P^*)}) = 0,$$

so $w = 0$ and $0 \in \partial H_{\text{dscrt}}(P^*)$.

On the other hand, if P^* is a critical point, then there exists scalars λ_{ij} satisfying (12), such that

$$w = \sum_{j=1}^{N} \phi_j \sum_{k \in I(q_j,P)} 2\lambda_{kj} (q_j - p_k^*) \frac{\partial p_k}{\partial P} = 0,$$

which implies, for each $i \in \{1, \dots, n\}$

$$w_i = 2 \sum_{q_j \in V_i(P^*)} \lambda_{ij} \phi_j (q_j - p_i^*) = 0,$$

Solve this linear equation, we obtain

$$p_i^* = (C_{\text{dscrt}})_{V_i(P^*)}, \qquad i \in \{1, \dots, n\}.$$

3.2 Continuous and discrete coverage control

Based on the expressions obtained in the previous subsection, it is possible to design motion coordination algorithms for the robots $p_1, \dots, p_n$. We call *continuous* and *discrete coverage control* the problem maximizing the multi-center functions $\mathcal{H}$ and $\mathcal{H}_{\mathrm{dscrt}}$, respectively. The continuous problem is studied in [18]. We simply impose that the locations $p_1, \dots, p_n$ follow a gradient ascent law defined over the set $Q^n \setminus \mathcal{S}_{\mathrm{coinc}}$. Formally, we set

$$u_i = k_{\mathrm{prop}} \frac{\partial \mathcal{H}}{\partial p_i}(P), \tag{13}$$

where k_{prop} is a positive gain. Note that this law is distributed in the sense that each robot only needs information about its Voronoi cell in order to compute its control.

For discrete coverage control, we adopt the following discontinuous control law, for each robot $i \in \{1, \dots, n\}$

$$u_i = k_{\mathrm{prop}}\, X_i(P), \tag{14}$$

where $X_i : Q^n \to \mathbb{R}^2$ is defined as

$$X_i(P) = \sum_{q_j \in V_i(P)} \frac{\phi_j}{\mathrm{card}(q_j, P)} \frac{\partial}{\partial p_i} f(\|q_j - p_i\|).$$

Note that X_i is continuous at $P \in Q^n \setminus \mathcal{S}_{\mathrm{coinc}} \cup \mathcal{S}_{\mathrm{equid}}$, and satisfies

$$X_i(P) = \frac{\partial \mathcal{H}_{\mathrm{dscrt}}}{\partial p_i}(P).$$

Like control law (13), the discontinuous control law (14) is also distributed. Define the vector field $X = [X_1, X_2, \dots, X_n]^T$, we have

$$\dot{P} = k_{\mathrm{prop}}\, X(P). \tag{15}$$

Since $X(P)$ is discontinuous at $P \in \mathcal{S}_{\mathrm{coinc}} \cup \mathcal{S}_{\mathrm{equid}}$, we understand the solution of this equation in the Filippov sense following [27], and the existence of Filippov solution is guaranteed. We then investigate the properties of the solution and analysis the convergence of (13) and (14).

Proposition 3.6 (Continuous coverage control; [18, 19])**.** *For the closed-loop systems induced by equation* (13) *starting at $P_0 \in Q^n \setminus \mathcal{S}_{\mathrm{coinc}}$, the agents location converges asymptotically to the set of critical points of $\mathcal{H}$.*

Proposition 3.7 (Discrete coverage control)**.** *For the closed-loop systems induced by equation* (14) *starting at $P_0 \in Q^n \setminus \mathcal{S}_{\mathrm{coinc}}$, the agents location converges asymptotically to the set of critical points of $\mathcal{H}_{\mathrm{dscrt}}$.*

Proof. Note that

$$K[k_{\mathrm{prop}}\, X](P) = k_{\mathrm{prop}} \partial \mathcal{H}_{\mathrm{dscrt}}(P).$$

Given this property, the following proof is essentially the same as the proof of Proposition 2.9 in [28]. We refer the interested reader to [28] for technical details.

3.3 Discretizing continuous settings

In this section we discuss the relationship between the discretization of continuous locational optimization problems and discrete locational optimization problems.

As before, let Q be a convex polygon in $\mathbb{R}^2$ including its interior, and let $\phi : \mathbb{R}^2 \to \overline{\mathbb{R}}_+$ be a bounded and measurable function whose support is Q. We shall consider a sequence of pointsets $\{q_1^k, \dots, q_{N_k}^k\}_{k \in \mathbb{N}} \subset \mathbb{R}^2$ and of nonnegative weights $\{\phi_1^k, \dots, \phi_{N_k}^k\}_{k \in \mathbb{N}}$. Accordingly, we can define a

sequence of discrete multi-center functions $\mathcal{H}^k_{\text{dscrt}}$, for $k \in \mathbb{N}$. The sequence $\{q^k_1, \dots, q^k_{N_k}\}_{k\in\mathbb{N}} \subset \mathbb{R}^2$ is *dense*[1] in Q if, for all $q \in Q$,

$$\lim_{k\to+\infty} \min\{\|q - z\| \mid z \in \{q^k_1, \dots, q^k_{N_k}\}\} = 0.$$

Given a pointset $q_1, \dots, q_N$, let $V(q_1, \dots, q_N)$ denote the Voronoi partition it generates and define the associated weights

$$\phi_j = \int_{V_j(q_1,\dots,q_N)} \phi(q)dq. \tag{16}$$

Proposition 3.8 (Consistent discretization)**.** *Assume that f is continuous almost everywhere, that the sequence $\{q^k_1, \dots, q^k_{N_k}\}_{k\in\mathbb{N}} \subset \mathbb{R}^2$ is dense in Q, and that the sequence of weights are defined according to* (16)*. Then $\{\mathcal{H}^k_{\text{dscrt}}\}_{k\in\mathbb{N}}$ converges pointwise to $\mathcal{H}$, that is, for all $P \in Q^n$,*

$$\lim_{k\to+\infty} \mathcal{H}^k_{\text{dscrt}}(P) = \mathcal{H}(P).$$

Additionally, if f is continuously differentiable, then for $P \in Q^n \setminus \mathcal{S}_{\text{coinc}}$ and each $i \in \{1, \dots, n\}$, any sequence $x_k \in \partial_i \mathcal{H}^k_{\text{dscrt}}(P)$, $k \in \mathbb{N}$, satisfies

$$\lim_{k\to+\infty} x_k = \frac{\partial \mathcal{H}}{\partial p_i}(P).$$

Proof. For $k \in \mathbb{N}$, given the pointset $\{q^k_1, \dots, q^k_{N_k}\}$, we define the projection $\text{proj}_k : Q \to \{q^k_1, \dots, q^k_{N_k}\}$ by

$$\text{proj}_k(q) = \text{argmin}\{\|q - z\| \mid z \in \{q^k_1, \dots, q^k_{N_k}\}\}.$$

Because of the vanishing dispersion property, we know that, for all $q \in Q$,

$$\lim_{k\to+\infty} \text{proj}_k(q) = q. \tag{17}$$

Therefore, we compute

$$\begin{aligned}
\mathcal{H}^k_{\text{dscrt}}(P) &= \sum_{j=1}^{N_k} \max_{i\in\{1,\dots,n\}} f(\|q^k_j - p_i\|) \int_{V_j(q^k_1,\dots,q^k_{N_k})} \phi(q)dq \\
&= \sum_{j=1}^{N_k} \int_{V_j(q^k_1,\dots,q^k_{N_k})} \max_{i\in\{1,\dots,n\}} f(\|q^k_j - p_i\|)\phi(q)dq \\
&= \int_Q \max_{i\in\{1,\dots,n\}} f(\|\text{proj}_k(q) - p_i\|)\phi(q)dq.
\end{aligned}$$

Because f is continuous almost everywhere, we have

$$\begin{aligned}
&\lim_{k\to+\infty} \mathcal{H}^k_{\text{dscrt}}(P) \\
&\quad= \lim_{k\to+\infty} \int_Q \max_{i\in\{1,\dots,n\}} f(\|\text{proj}_{N_k}(q) - p_i\|)\phi(q)dq \\
&\quad= \int_Q \max_{i\in\{1,\dots,n\}} f(\|\lim_{k\to+\infty} \text{proj}_{N_k}(q) - p_i\|)\phi(q)dq \\
&\quad= \int_Q \max_{i\in\{1,\dots,n\}} f(\|q - p_i\|)\phi(q)dq = \mathcal{H}(P).
\end{aligned}$$

[1] This is equivalent to asking that the sequence has *vanishing dispersion*; the dispersion of a pointset $\{q_1, \dots, q_N\}$ in the compact set Q is $\max_{q\in Q} \min_{z\in\{q_1,\dots,q_N\}} \|q - z\|$.

Define

$$\partial_i^* \mathcal{H}_{\text{dscrt}}^k(P) = \sum_{q_j^k \in \overset{\circ}{V}_i(P)} \phi_j \frac{\partial}{\partial p_i} f(\|q_j^k - p_i\|) + \sum_{q_j^k \in \partial V_i(P)} \phi_j \operatorname{co}\left\{ \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \frac{\partial}{\partial p_i} f(\|q_j^k - p_i\|) \right\}.$$

By Corollary 3.4, if f is differentiable, then $\partial_i \mathcal{H}_{\text{dscrt}}^k(P) \subset \partial_i^* \mathcal{H}_{\text{dscrt}}^k(P)$. Suppose $x_k^* \in \partial_i^* \mathcal{H}_{\text{dscrt}}^k(P)$, then there exists scalars $\lambda_{ij} \in [0,1]$, such that

$$x_k^* = \sum_{q_j^k \in \overset{\circ}{V}_i(P)} \phi_j \frac{\partial}{\partial p_i} f(\|q_j^k - p_i\|) + \sum_{q_j^k \in \partial V_i(P)} \lambda_{ij} \phi_j \frac{\partial}{\partial p_i} f(\|q_j^k - p_i\|). \tag{18}$$

Substitute (16) into (18), we obtain

$$\begin{aligned} x_k^* = & \sum_{q_j^k \in \overset{\circ}{V}_i(P)} \int_{V_j(q_1^k, \ldots, q_N^k)} \phi(q) \frac{\partial}{\partial p_i} f(\|q_j^k - p_i\|) dq \\ & + \sum_{q_j^k \in \partial V_i(P)} \int_{V_j(q_1^k, \ldots, q_N^k)} \lambda_{ij} \phi(q) \frac{\partial}{\partial p_i} f(\|q_j^k - p_i\|) dq. \end{aligned}$$

Since f is continuously differentiable, so for $P \in Q^n \setminus \mathcal{S}_{\text{coinc}}$, we have

$$\lim_{k \to +\infty} x_k^* = \int_{\overset{\circ}{V}_i(P)} \phi(q) \frac{\partial}{\partial p_i} f(\|q - p_i\|) dq + \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \frac{\partial \mathcal{H}}{\partial p_i}(P).$$

Hence,

$$\lim_{k \to +\infty} x_k = \frac{\partial \mathcal{H}}{\partial p_i}(P).$$

3.4 The relationship between discrete coverage and averaging over switching acyclic digraphs

As above, let Q be a convex polygon, let $\{p_1, \ldots, p_n\} \subset Q$ be the position of n robots, let $\{q_1, \ldots, q_N\} \subset Q$ be N fixed points in Q with corresponding nonnegative weights $\{\phi_1, \ldots, \phi_N\}$, and let $I(q_j, P)$ be the set of indices k for which $\|q_j - p_k\| = \min_{i \in \{1,\ldots,n\}} \|q_j - p_i\|$. We begin by defining a useful digraph and a useful set of digraphs.

A *discrete Voronoi graph* $\mathcal{G}_{\text{dscrt-Vor}}$ is a digraph with $(n+N)$ vertices $\{p_1, \ldots, p_n, q_1, \ldots, q_N\}$, with N directed edges

$$\{(p_i, q_j) |\text{ for each } j \in \{1, \ldots, N\}, \text{ pick one and only one } i \in I(q_j, P)\},$$

and with corresponding edge weights ϕ_j, for all $j \in \{1, \ldots, N\}$. We illustrate this graph in Figure 3. With our definition, it is possible for one vertex set to generate multiple discrete Voronoi graphs. We will denote the nodes of $\mathcal{G}_{\text{dscrt-Vor}}$ by $Z = (z_1, \ldots, z_{n+N}) \in (\mathbb{R}^2)^{n+N}$, the weights by $a_{\alpha\beta}$, for $\alpha, \beta \in \{1, \ldots, n+N\}$, with the understanding that:

$$z_\alpha = \begin{cases} p_\alpha, & \text{if } \alpha \in \{1, \ldots, n\}, \\ q_{\alpha - n}, & \text{otherwise,} \end{cases}$$

and that the only non-vanishing weights are $a_{\alpha\beta} = \phi_j$ when $\beta = n + j$, for $j \in \{1, \ldots, N\}$, and when $\alpha \in \{1, \ldots, n\}$ corresponds to the robot p_α closest to q_j and (p_α, q_j) is a directed edge of the

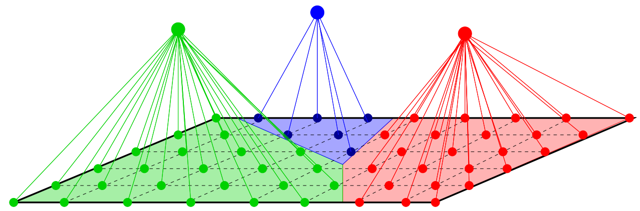

Figure 3: The discrete Voronoi graph over 3 robots and 6×9 grid points. This illustration is to be compared with the Voronoi partition illustrated in Figure 2. The edges have top/down direction.

graph $\mathcal{G}_{\text{dscrt-Vor}}$. Note that $\mathcal{G}_{\text{dscrt-Vor}}$ depends upon Z. Since $\{q_1, \dots, q_N\} \subset Q$ are fixed, when we need to emphasize this dependence, we will simply denote it as $\mathcal{G}_{\text{dscrt-Vor}}(P)$.

Let us now define a set of digraphs of which the discrete Voronoi graphs are examples. Let $F(N, n)$ be the set of functions from $\{1, \dots, N\}$ to $\{1, \dots, n\}$. Roughly speaking, a function in $F(N, n)$ assigns to each point q_j, $j \in \{1, \dots, N\}$, a robot p_i, $i \in \{1, \dots, n\}$. Given $h \in F(N, n)$, let $\mathcal{G}_h$ be the digraph with $(n + N)$ vertices $\{p_1, \dots, p_n, q_1, \dots, q_N\}$, with N directed edges

$$\{(p_{h(j)}, q_j)\}_{j \in \{1,\dots,N\}},$$

and corresponding edge weights ϕ_j, $j \in \{1, \dots, N\}$. With these notations, it holds that $\mathcal{G}_{\text{dscrt-Vor}}(P) = \mathcal{G}_{h^*(\cdot,P)}$ with any function $h^* : \{1, \dots, N\} \times Q^n \to \{1, \dots, n\}$ which satisfies

$$h^*(j, P) \in \operatorname{argmin}\{\|q_j - p_i\| \mid i \in \{1, \dots, n\}\}.$$

Let us state a useful observation about these digraphs.

Lemma 3.9. *The set of digraphs $\mathcal{G}_h$, $h \in F(N, n)$, is a set of acyclic digraphs with unit depth, i.e., it is a subset of $\mathcal{S}_{n+N,1}$ (see definition in Subsection 2.3). Moreover, $\cup_{h \in F(N,n)} \mathcal{G}_h$ is an acyclic digraph with unit depth, i.e., $\cup_{h \in F(N,n)} \mathcal{G}_h \in \mathcal{S}_{n+N,1}$.*

For $h \in F(N, n)$, let us study appropriate disagreement functions for the digraph $\mathcal{G}_h$. We define the function $\Phi_{\mathcal{G}_h} : (\mathbb{R}^2)^{n+N} \to \mathbb{R}$ by

$$\begin{aligned} \Phi_{\mathcal{G}_h}(Z)|_{Z=(p_1,\dots,p_n,q_1,\dots,q_N)} &= \frac{1}{2} \sum_{\alpha,\beta=1}^{n+N} a_{\alpha\beta} \|z_\alpha - z_\beta\|^2 \\ &= \frac{1}{2} \sum_{j=1}^{N} \phi_j \|q_j - p_{h(j)}\|^2, \end{aligned}$$

because the weights $a_{\alpha\beta}$, $\alpha, \beta \in \{1, \dots, n + N\}$ of the digraph $\mathcal{G}_h$ all vanish except for $a_{h(j),j} = \phi_j$, $j \in \{1, \dots, N\}$.

We are now ready to state the main result of this section. The proof of the following theorem is based on simple book-keeping and is therefore omitted.

Theorem 3.10 (Discrete coverage control and averaging)**.** *The following statements hold:*

(i) The discrete multi-center function $\mathcal{H}_{\text{dscrt}}$ with $f(x) = -x^2$, and the minimum disagreement function over the set of digraphs $\mathcal{G}_h$, $h \in F(N, n)$, satisfy

$$\begin{aligned} -\frac{1}{2}\mathcal{H}_{\text{dscrt}}(P) &= \frac{1}{2} \sum_{j=1}^{N} \min_{i \in \{1,\dots,n\}} \phi_j \|q_j - p_i\|^2 \\ &= \frac{1}{2} \sum_{j=1}^{N} \phi_j \|q_j - p_{h^*(j)}\|^2 \\ &= \Phi_{\mathcal{G}_{\text{dscrt-Vor}}}(p_1, \dots, p_n, q_1, \dots, q_N) \\ &= \min_{h \in F(N,n)} \Phi_{\mathcal{G}_h}(p_1, \dots, p_n, q_1, \dots, q_N). \end{aligned}$$

(ii) For $P \notin \mathcal{S}_{\text{coinc}} \cup \mathcal{S}_{\text{equid}}$, the discrete coverage control law for $f(x) = -x^2$ and the averaging protocol over the discrete Voronoi digraph satisfy, for $i \in \{1, \dots, n\}$,

$$\frac{1}{2}\frac{\partial \mathcal{H}_{\text{dscrt}}}{\partial p_i}(P) = \sum_{q_j \in V_i(P)} \phi_j(q_j - p_i) = \sum_{\beta=1}^{n+N} a_{\alpha\beta}(z_\beta - z_\alpha),$$

where z_α and $a_{\alpha\beta}$, $\alpha, \beta \in \{1, \dots, n+N\}$, are nodes and weights of $\mathcal{G}_{\text{dscrt-Vor}}$. Accordingly, the discontinuous coverage control system (15), for $f(x) = -x^2$, and the averaging system (7) over the set of digraphs $\mathcal{G}_h$, $h \in F(N, n)$ satisfy, for $i \in \{1, \dots, n\}$,

$$\frac{1}{2} K[X_i](P) = K[Y_i](Z),$$

where $Z = (p_1, \dots, p_n, q_1, \dots, q_N)$, X_i and Y_i are the i^{th} 2-dimensional block component of X and Y, respectively.

(iii) Any $P^* \in Q^n$ is an equilibrium of the discrete coverage control system with $f(x) = -x^2$ if and only if $Z^* = (p_1^*, \dots, p_n^*, q_1, \dots, q_N)$ is an equilibrium of system (7) over the set of digraphs $\mathcal{G}_h$, $h \in F(N, n)$, that is:

$$\begin{aligned} &\forall j \in \{1, \dots, N\},\ \exists \lambda_{ij} \text{ as in (12), such that } p_i^* = (C_{\text{dscrt}})_{V_i(P^*)},\ \forall i \in \{1, \dots, n\}, \\ \iff \quad &\exists \mu_k \geq 0 \text{ and } \sum_k \mu_k = 1, \text{ such that } Z^* \in \ker\Big(\sum_k \mu_k L(\mathcal{G}^k_{\text{dscrt-Vor}}(Z^*))\Big), \end{aligned}$$

where $\{\mathcal{G}^k_{\text{dscrt-Vor}}(Z^*)\}_k$ are all possible discrete Voronoi graphs generated by Z^*.

(iv) Given any initial position of robots $P_0 \in Q^n$, the evolution of the discrete coverage control system (15) and the evolution of the averaging system (7) under the switching signal $\sigma : Q^n \to \{\mathcal{G}_h \mid h \in F(N, n)\}$ defined by $\sigma(P) = \mathcal{G}_{\text{dscrt-Vor}}(Z)$ are identical in the Filippov sense and, therefore, the two systems will converge to the same set of equilibrium placement of robots, as described in (iii).

3.5 Numerical simulations

To illustrate the performance of the discrete coverage law as stated in Proposition 3.7 and to illustrate the accuracy of the discretization process, as analyzed in Proposition 3.8, we include some simulation results. The algorithms are implemented in `Matlab` as a single centralized program. As expected, the simulations for the discrete coverage law are computationally intensive with the increase in the resolution of the grid. We illustrate the performance of the closed-loop systems in Figures 4, 5, 6 and 7.

4 Conclusions

We have studied averaging protocols over fixed and controlled-switching acyclic digraphs, and characterized their asymptotic convergence properties. We have also discussed continuous and discrete multi-center locational optimization functions, and distributed control laws that optimize them. The main result of the paper shows how these two sets of problems are intimately related: discrete coverage control laws are indeed averaging protocols over acyclic digraphs. As a consequence of our analysis, it may be argued that the coverage control problem and the consensus problem are both special cases of a general class of distributed optimization problems.

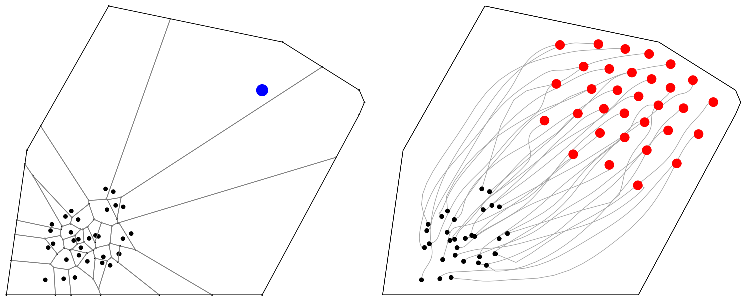

Figure 4: Continuous coverage law for 32 agents on a convex polygonal environment, with density function $\phi = \exp(5.(-x^2 - y^2))$ centered about the gray point in the figure. The control gain in (13) is $k_{\text{prop}} = 1$ for all the vehicles. The left (respectively, right) figure illustrates the initial (respectively, final) locations and Voronoi partition. The central figure illustrates the gradient descent flow. Figure taken from [18].

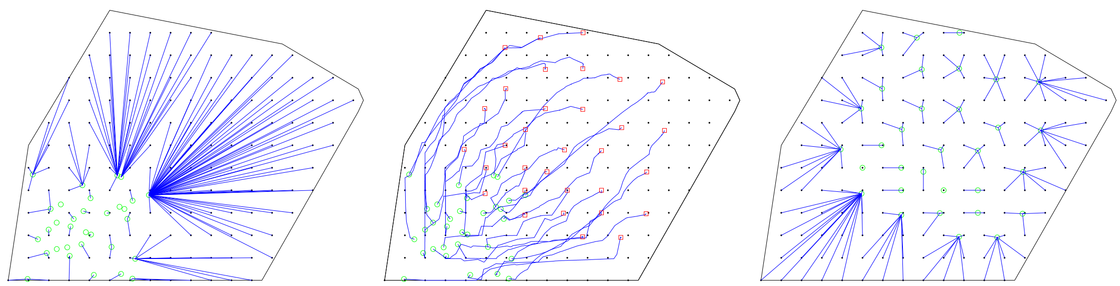

Figure 5: Simulation of discrete coverage law with 159 grid points.

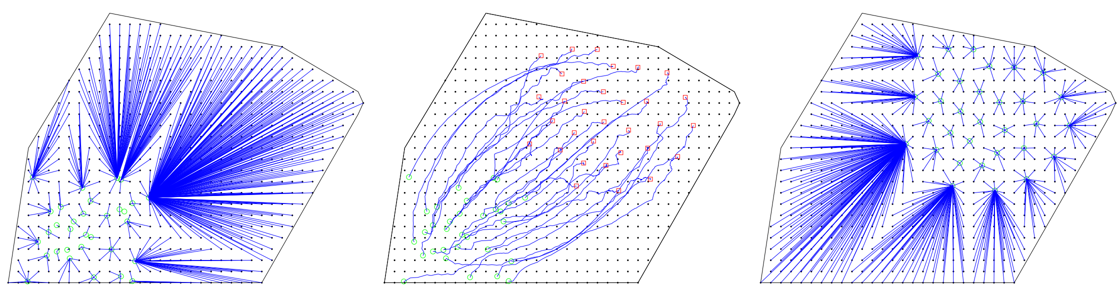

Figure 6: Simulation of discrete coverage law with 622 grid points.

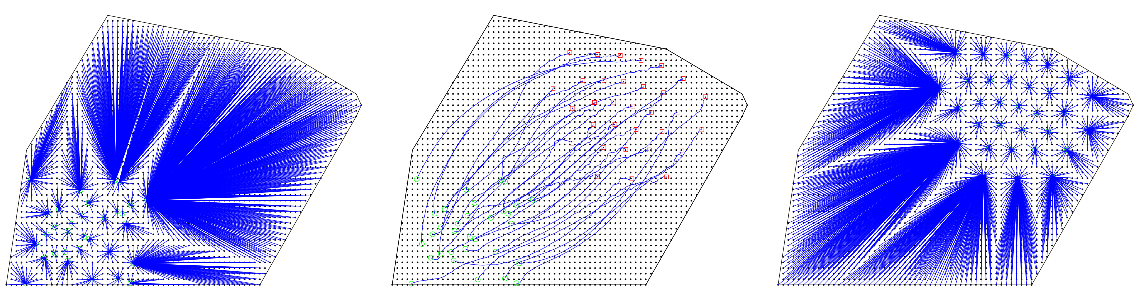

Figure 7: Simulation of discrete coverage law with 2465 grid points.

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A Basic ideas on nonsmooth analysis

In this appendix we review some properties in basic calculus of generalized gradient of functions [29].

Proposition A.1 (Scalar multiples)**.** *If $f : \mathbb{R}^M \to \mathbb{R}$ is locally Lipschitz at $x \in \mathbb{R}^M$, then, for any scalar s, sf is locally Lipschitz at x and*

$$\partial(sf)(x) = s\partial f(x).$$

Proposition A.2 (Finite sums)**.** *If $f_i : \mathbb{R}^M \to \mathbb{R}$, $i \in \{1, \dots, m\}$, are locally Lipschitz at $x \in \mathbb{R}^M$, then, for any scalars $\{s_1, \dots, s_m\}$, $\sum_{i=1}^m s_i f_i$ is locally Lipschitz and*

$$\partial\Big(\sum_{i=1}^{m} s_i f_i\Big)(x) \subset \sum_{i=1}^{m} s_i \partial f_i(x),$$

where equality holds if each f_i is regular at x and each s_i is nonnegative.

Proposition A.3. *Let $f_k : \mathbb{R}^M \to \mathbb{R}$, $k \in \{1, \dots, m\}$ be locally Lipschitz functions at $x \in \mathbb{R}^M$ and let $f(x') = \max\{f_k(x') \mid k \in \{1, \dots, m\}\}$. Then,*

(i) f is locally Lipschitz at x,

(ii) if $I(x')$ denotes the set of indexes k for which $f_k(x') = f(x')$, we have

$$\partial f(x) \subset \mathrm{co}\{\partial f_i(x) \mid i \in I(x)\}\,,$$

and if f_i, $i \in I(x)$, is regular at x, then equality holds and f is regular at x.

B Stability analysis via nonsmooth Lyapunov functions

Throughout the paper, we define the solutions of differential equations with discontinuous right-hand sides in terms of differential inclusions [27]. Let $F : \mathbb{R}^N \to 2^{\mathbb{R}^N}$ be a set-valued map. Consider the differential inclusion

$$\dot{x} \in F(x)\,. \tag{19}$$

A solution to this equation on an interval $[t_0, t_1] \subset \mathbb{R}$ is defined as an absolutely continuous function $x : [t_0, t_1] \to \mathbb{R}^N$ such that $\dot{x}(t) \in F(x(t))$ for almost all $t \in [t_0, t_1]$. Given $x_0 \in \mathbb{R}^N$, the existence of at least a solution with initial condition x_0 is guaranteed by the following lemma.

Lemma B.1. *Let the mapping F be upper semicontinuous with nonempty, compact and convex values. Then, given $x_0 \in \mathbb{R}^N$, there exists a local solution of* (19) *with initial condition x_0.*

Now, consider the differential equation

$$\dot{x}(t) = X(x(t))\,, \tag{20}$$

where $X : \mathbb{R}^N \to \mathbb{R}^N$ is measurable and essentially locally bounded. Here, we understand the solution of this equation in the Filippov sense, which we define in the following. For each $x \in \mathbb{R}^N$, consider the set

$$K[X](x) = \bigcap_{\delta > 0} \bigcap_{\mu(S) = 0} \mathrm{co}\{X(B_N(x, \delta) \setminus S)\}\,,$$

where μ denotes the usual Lebesgue measure in $\mathbb{R}^N$. Alternatively, one can show [30] that there exists a set S_X of measure zero such that

$$K[X](x) = \mathrm{co}\{\lim_{i \to +\infty} X(x_i) \mid x_i \to x\,,\ x_i \notin S \cup S_X\}\,,$$

where S is any set of measure zero. A Filippov solution of (20) on an interval $[t_0, t_1] \subset \mathbb{R}$ is defined as a solution of the differential inclusion $\dot{x} \in K[X](x)$. Since the multivalued mapping $K[X] : \mathbb{R}^N \to 2^{\mathbb{R}^N}$ is upper semicontinuous with nonempty, compact, convex values and locally bounded (cf. [27]), the existence of Filippov solutions of (20) is guaranteed by Lemma B.1.

Given a locally Lipschitz function $f : \mathbb{R}^N \to \mathbb{R}$, the *set-valued Lie derivative of f with respect to X* at x is defined as

$$\widetilde{\mathcal{L}}_X f(x) = \{a \in \mathbb{R} \mid \exists v \in K[X](x) \text{ such that } \zeta \cdot v = a\,,\ \forall \zeta \in \partial f(x)\}\,.$$

For each $x \in \mathbb{R}^N$, $\widetilde{\mathcal{L}}_X f(x)$ is a closed and bounded interval in $\mathbb{R}$, possibly empty. If f is continuously differentiable at x, then $\widetilde{\mathcal{L}}_X f(x) = \{df \cdot v \mid v \in K[X](x)\}$. If, in addition, X is continuous at x, then $\widetilde{\mathcal{L}}_X f(x)$ corresponds to the singleton $\{\mathcal{L}_X f(x)\}$, the usual Lie derivative of f in the direction of X at x.

The following result is a generalization of LaSalle Invariance Principle for differential equations of the form (20) with nonsmooth Lyapunov functions. The formulation is taken from [31], and slightly generalizes the one presented in [32].

Theorem B.2 (LaSalle principle)**.** *Let $f : \mathbb{R}^N \to \mathbb{R}$ be a locally Lipschitz and regular function. Let $x_0 \in \mathbb{R}^N$ and let $f^{-1}(\leq f(x_0), x_0)$ be the connected component of $\{x \in \mathbb{R}^N \mid f(x) \leq f(x_0)\}$ containing x_0. Assume the set $f^{-1}(\leq f(x_0), x_0)$ is bounded and assume either* $\max \widetilde{\mathcal{L}}_X f(x) \leq 0$ *or $\widetilde{\mathcal{L}}_X f(x) = \emptyset$ for all $x \in f^{-1}(\leq f(x_0), x_0)$. Then $f^{-1}(\leq f(x_0), x_0)$ is strongly invariant for* (20)*. Let*

$$Z_{X,f} = \{x \in \mathbb{R}^N \mid 0 \in \widetilde{\mathcal{L}}_X f(x)\} .$$

Then, any solution $x : [t_0, +\infty) \to \mathbb{R}^N$ of (20) *starting from x_0 converges to the largest weakly invariant set M contained in $\overline{Z}_{X,f} \cap f^{-1}(\leq f(x_0), x_0)$. Furthermore, if the set M is a finite collection of points, then the limit of all solutions starting at x_0 exists and equals one of them.*